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On Uniform Asymptotic Expansion of Definite Integrals*

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1. INTRODUCTION

In this paper we are concerned with the asymptotic expansion of integrals of the form

$$I(\lambda) = \int_0^b t^{r-1} q(t)\, e^{-\lambda p(t)}\, dt, \tag{1.1}$$

where $r > 0$ is a fixed number and λ is a large positive parameter. The range of integration is real and may be finite or infinite. We assume that (i) $q(t)$ is analytic in some neighborhood of the path of integration;

(ii) $p(t)$ is analytic in some neighborhood of $[-\alpha, b)$, where $\alpha \geqslant 0$;

(iii) $p'(-\alpha) = 0$, $p''(-\alpha) > 0$, and $p'(t) > 0$ for $t > -\alpha$.

When α is a fixed number, it is well known (see, for example, [9]) that asymptotic expansions can be found by an extension of the method of Laplace. However, this method gives different kinds of expansions for different values of α, depending on whether $\alpha = 0$ or $\alpha > 0$. Our objective here is to obtain a form of expansion which will hold uniformly for α restricted to a fixed interval, say, $0 \leqslant \alpha \leqslant \alpha_0$. For earlier work on this problem see Bleistein [1].

2. REDUCTION TO A CANONICAL FORM

In the classical situation when α is a fixed number, the use of the substitution,

$$p(t) - p(0) = u^k, \tag{2.1}$$

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is often envisaged, where $k = 1$ or 2 depends on whether $\alpha > 0$ or $\alpha = 0$. The integral $I(\lambda)$ is then reduced to a canonical representation of the form

$$e^{\lambda p(0)} I(\lambda) = \int_0^B u^{r-1} f(u)\, e^{-\lambda u^k}\, du, \tag{2.2}$$

where

$$B = \{p(b) - p(0)\}^{1/k}, \tag{2.3}$$

and

$$u^{r-1} f(u) = t^{r-1} q(t)(dt/du). \tag{2.4}$$

The determination of the asymptotic behavior of $I(\lambda)$ given by (1.1), therefore, becomes equivalent to the determination of the asymptotic behavior of

$$J(\lambda) = \int_0^B u^{r-1} f(u)\, e^{-\lambda u^k}\, du. \tag{2.5}$$

If $\alpha > 0$ then $p'(0) > 0$. In this case we let $k = 1$ and

$$u = p(t) - p(0) = p'(0)\, t + \tfrac{1}{2} p''(0)\, t^2 + \cdots. \tag{2.6}$$

On the other hand, if $\alpha = 0$ then $p'(0) = 0$. In that case we let $k = 2$ and

$$u^2 = p(t) - p(0) = \tfrac{1}{2} p''(0)\, t^2 + (1/3!)\, p'''(0)\, t^3 + \cdots. \tag{2.7}$$

In both cases the inversion theorem of Burmann–Lagrange is applicable and will yield an expansion of the form

$$t = c_1 u + c_2 u^2 + c_3 u^3 + \cdots, \qquad c_1 \neq 0. \tag{2.8}$$

Substitution of (2.8) into (2.4) then gives

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots, \tag{2.9}$$

the coefficients a_s being expressible in terms of the Maclaurin coefficients of q and p.

If, in (2.5), we replace the upper limit by infinity, substitute (2.9), and integrate formally term by term, we obtain

$$I(\lambda) \sim e^{-\lambda p(0)} \sum_{m=0}^{\infty} \frac{a_m \Gamma(m + r)}{\lambda^{m+r}}, \qquad \text{if} \quad k = 1, \tag{2.10}$$

and

$$2I(\lambda) \sim e^{-\lambda p(0)} \sum_{m=0}^{\infty} \frac{a_m \Gamma\{(m + r)/2\}}{\lambda^{(m+r)/2}}, \qquad \text{if} \quad k = 2. \tag{2.11}$$

The justification of this formal process is provided by the well-known lemma of Watson, see Luke [4, Section 1.4]. Since there is a discontinuous change from an expansion in powers of λ^{-1} to an expansion in powers of $\lambda^{-1/2}$, any uniformly asymptotic expansion for $\alpha \geqslant 0$ must take account not only of these two types but also the transition between them.

Examining the conditions on $p(t)$ given in (iii), it is easy to see that the simplest example of such a function is provided by the polynomial $t^2/2 + \alpha t$. This suggests that instead of making the substitution

$$p(t) - p(0) = u \qquad \text{or} \qquad u^2, \tag{2.12}$$

we let

$$p(t) - p(0) = (u^2/2) + au, \tag{2.13}$$

where a is a parameter to be determined. In order for (2.13) to result in a single-valued analytic function $t = t(u)$, neither dt/du nor du/dt can vanish in the relevant regions. But

$$dt/du = (u + a)/p'(t), \tag{2.14}$$

and $p'(-\alpha) = 0$. Therefore we must make $t = -\alpha$ correspond to $u = -a$; i.e., we must choose

$$a = [2\{p(0) - p(-\alpha)\}]^{1/2}. \tag{2.15}$$

With this choice of a, $t = t(u)$, the solution of (2.13), is anlytic in a neighborhood of $u = 0$ and is monotonic on $[-\alpha, b)$.

We return to the integral (1.1) after making the substitution (2.13). The result is

$$I(\lambda) = e^{-\lambda p(0)} \int_0^B u^{r-1} f(u)\, e^{-\lambda[u^2/2+au]}\, du, \tag{2.16}$$

where

$$f(u) = (t/u)^{r-1}\, q(t)(dt/du) \tag{2.17}$$

and

$$B = \lim_{t\to b} \{2[p(t) - p(-\alpha)]\}^{1/2} - \{2[p(0) - p(-\alpha)]\}^{1/2}. \tag{2.18}$$

Since $t(u)$ is analytic near $u = 0$, $t(0) = 0$ and $t'(0) > 0$, the function $f(u)$ can again be expanded in the form (2.9)

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots, \tag{2.19}$$

where the series is converging in a neighborhood of $u = 0$.

The problem of finding the uniform asymptotic expansion of the integral

$I(\lambda)$, given by (1.1), is now reduced to that of finding the uniform asymptotic expansion of the integral

$$F(\lambda) = \int_0^\infty u^{r-1} f(u)\, e^{-\lambda[u^2/2+au]}\, du. \tag{2.20}$$

Since $f(u)$ is not required to be continuous, the seemingly more general integral

$$\int_0^B u^{r-1} f(u)\, e^{-\lambda[u^2/2+au]}\, du, \tag{2.21}$$

where B may or may not be finite, is not really a generalization of (2.20). This situation is covered by allowing $f(u)$ in (2.20) to satisfy $f(u) = 0$ for $u \geqslant B$.

3. Uniform Asymptotic Expansions

It is well known that Watson's lemma depends for its success on the basic integral

$$\int_0^\infty t^{\nu-1} e^{-\lambda t}\, dt = \Gamma(\nu)/\lambda^\nu, \qquad \nu > 0. \tag{3.1}$$

In our case, the corresponding integral is

$$\int_0^\infty t^{\nu-1} e^{-\lambda(t^2/2+\alpha t)}\, dt, \qquad \nu > 0, \tag{3.2}$$

which can be evaluated by means of the parabolic cylinder function [8] to be

$$\int_0^\infty u^{\nu-1} e^{-\lambda[u^2/2+\alpha u]}\, du = \frac{\Gamma(\nu)}{\lambda^{\nu/2}} D_{-\nu}(\lambda^{1/2}\alpha)\, e^{\lambda\alpha^2/4}. \tag{3.3}$$

The most important recursion formulae for the function $D_\nu(z)$ are

$$D_\nu'(z) + (z/2)\, D_\nu(z) - \nu D_{\nu-1}(z) = 0, \tag{3.4}$$

and

$$D_{\nu+1}(z) - z D_\nu(z) + \nu D_{\nu-1}(z) = 0. \tag{3.5}$$

These can be readily derived from their integral representations (see [8, p. 350]). The following lemma is essential in the proof of our main result.

Lemma. *For each $m \geqslant 1$ we have*

$$D_{-r-m}(z) = D_{-r}(z)\, P_m(z) + D'_{-r}(z)\, Q_{m-1}(z), \tag{3.6}$$

where $P_m(z)$ and $Q_{m-1}(z)$ are polynomials of the form

$$P_m(z) = \sum_{k=0}^{[m/2]} p_{m,k} z^{m-2k}, \tag{3.7}$$

$$Q_{m-1}(z) = \sum_{k=0}^{[m-1/2]} q_{m-1,k} z^{m-(2k+1)}. \tag{3.8}$$

The coefficients $p_{m,k}$ and $q_{m-1,k}$ can be successively determined from the recurrence relations

$$(r + m + 1)\, P_{m+2}(z) = P_m(z) - zP_{m+1}(z), \tag{3.9}$$

$$(r + m + 1)\, Q_{m+1}(z) = Q_{m-1}(z) - zQ_m(z), \tag{3.10}$$

with

$$\begin{aligned} P_1(z) &= -z/2r, & (r+1)\, P_2(z) &= 1 + (z^2/2r), \\ Q_0(z) &= -1/r, & (r+1)\, Q_1(z) &= z/r. \end{aligned} \tag{3.11}$$

Proof. We proceed by induction on m. From (3.4)

$$(-r)\, D_{-r-1}(z) = D'_{-r}(z) + (z/2)\, D_{-r}(z), \tag{3.12}$$

and hence (3.6) holds for $m = 1$. So we assume that the result is true for all $m \leqslant s$. It then follows from (3.5) that

$$(r + s)\, D_{-r-(s+1)}(z) = D_{-r-(s-1)}(z) - zD_{-r-s}(z). \tag{3.13}$$

Using our inductive hypothesis, we may express $D_{-r-(s-1)}(z)$ and $D_{-r-s}(z)$ in the form of (3.6). Thus,

$$D_{-r-(s+1)}(z) = D_{-r}(z)\, P_{s+1}(z) + D'_{-r}(z)\, Q_s(z), \tag{3.14}$$

where $P_{s+1}(z)$ and $Q_s(z)$ are given in (3.9) and (3.10), respectively. Since $P_{s-1}(z)$ and $P_s(z)$ are assumed to be of the form (3.7), it is easy to see from (3.9) that $P_{s+1}(z)$ must also be of this form. The same argument applies to $Q_s(z)$. Furthermore, since the polynomials P_1, P_2, Q_0, and Q_1, as given in (3.11), can be obtained directly from (3.12) and (3.13), the recursion formulae (3.9) and (3.10) can be used to successively determine the coefficients of $P_m(z)$ and $Q_{m-1}(z)$ for $m \geqslant 2$. This completes the proof of the lemma.

Remark. The above result is also valid when $m = 0$ if we agree to set $Q_{-1}(z) = 0$ and $P_0(z) = 1$.

MAIN THEOREM. *Consider the integral*

$$F(\lambda) = \int_0^\infty u^{r-1} f(u)\, e^{-\lambda[u^2/2+\alpha u]}\, du, \tag{3.16}$$

where $r > 0$, $\alpha \geqslant 0$ *and* $\lambda \to +\infty$. *If* $f(u)$ *can be expanded in the form*

$$f(u) = \sum_{m=0}^{\infty} a_m u^m, \qquad |u| \leqslant R, \tag{3.17}$$

and if there exists positive constants K_1 *and* K_2 *such that*

$$|f(u)| \leqslant K_1 e^{K_2 u^2}, \qquad u \geqslant R; \tag{3.18}$$

then for any fixed $N \geqslant 0$ *we have, uniformly in* α,

$$e^{-\lambda\alpha^2/4} F(\lambda) = \frac{D_{-r}(\lambda^{1/2}\alpha)}{\lambda^{r/2}} \left[\sum_{k=0}^{N} \frac{\zeta_k(\alpha)}{\lambda^k} + 0\left(\frac{1}{\lambda^{N+1}}\right)\right]$$

$$+ \frac{D'_{-r}(\lambda^{1/2}\alpha)}{\lambda^{(r+1)/2}} \left[\sum_{k=0}^{N} \frac{\eta_k(\alpha)}{\lambda^k} + 0\left(\frac{1}{\lambda^{N+1}}\right)\right]$$

where $\zeta_k(\alpha)$ *and* $\eta_k(a)$ *are analytic functions of* α *given by*

$$\zeta_k(\alpha) = \sum_{m \geqslant 2k}^{2N+2} a_m \Gamma(r+m)\, p_{m,k} \alpha^{m-2k}, \tag{3.20}$$

$$\eta_k(\alpha) = \sum_{m \geqslant 2k+1}^{2N+2} a_m \Gamma(r+m)\, q_{m-1,k} \alpha^{m-2k-1}, \tag{3.21}$$

the p's and q's being the coefficients of $P_m(z)$ *and* $Q_{m-1}(z)$ *given by* (3.7) *and* (3.8).

Proof. For any integer $N \geqslant 0$, we set

$$f(u) = \sum_{m=0}^{2N+2} a_m u^m + R_{2N+2}\,. \tag{3.22}$$

Then

$$F(\lambda) = \sum_{m=0}^{2N+2} a_m \int_0^\infty u^{r+m-1} e^{-\lambda[u^2/2+\alpha u]}\, du + E_{2N+2}\,, \tag{3.23}$$

where

$$E_{2N+2} = \int_0^\infty R_{2N+2} u^{r-1} e^{-\lambda[u^2/2+\alpha u]}\, du. \tag{3.24}$$

From (3.3) we have

$$e^{-\lambda\alpha^2/4}F(\lambda) = \sum_{m=0}^{2N+2} \frac{a_m\Gamma(m+r)}{\lambda^{(r+m)/2}} D_{-r-m}(\lambda^{1/2}\alpha) + e^{-\lambda\alpha^2/4}E_{2N+2}\,. \tag{3.25}$$

By Lemma 1 the finite sum in (3.25) can be rewritten as

$$\frac{D_{-r}(\lambda^{1/2}\alpha)}{\lambda^{r/2}} \sum_{m=0}^{2N+2} \sum_{k=0}^{[m/2]} \frac{a_m\Gamma(r+m)}{\lambda^k} p_{m,k}\alpha^{m-2k} + \frac{D'_{-r}(\lambda^{1/2}\alpha)}{\lambda^{(r+1)/2}} \sum_{m=1}^{2N+2} \sum_{k=0}^{[(m-1)/2]} \frac{a_m\Gamma(r+m)}{\lambda^k} q_{m-1,k}\alpha^{m-2k-1}.$$

An interchange of the summation signs then yields

$$\sum_{m=0}^{2N+2} \frac{a_m\Gamma(m+r)}{\lambda^{(m+r)/2}} D_{-r-m}(\lambda^{1/2}\alpha) = \frac{D_{-r}(\lambda^{1/2}\alpha)}{\lambda^{r/2}} \sum_{k=0}^{N+1} \frac{\zeta_k(\alpha)}{\lambda^k} + \frac{D'_{-r}(\lambda^{1/2}\alpha)}{\lambda^{(r+1)/2}} \sum_{k=0}^{N} \frac{\eta_k(\alpha)}{\lambda^k}, \tag{3.27}$$

where the ζ_k's and η_k's are given by (3.20) and (3.21). Coupling this result together with (3.25), we have

$$e^{-\lambda\alpha^2/4}F(\lambda) = \frac{D_{-r}(\lambda^{1/2}\alpha)}{\lambda^{r/2}} \sum_{k=0}^{N+1} \frac{\zeta_k(\alpha)}{\lambda^k} + \frac{D'_{-r}(\lambda^{1/2}\alpha)}{\lambda^{(r+\nu)/2}} \sum_{k=0}^{N} \frac{\eta_k(\alpha)}{\lambda^k} + e^{-\lambda\alpha^2/4}E_{2N+2}\,. \tag{3.28}$$

To complete the proof, an estimate of E_{2N+2} must be made. From (3.17), we have

$$|R_{2N+2}| \leqslant K_3 u^{2N+3}, \tag{3.29}$$

for $0 \leqslant u \leqslant R$. With (3.18)

$$|R_{2N+2}| \leqslant K_4 u^{2N+3} e^{K_2 u^2}, \tag{3.30}$$

for all $u \geqslant 0$, whether $u \leqslant R$ or $u \geqslant R$. Since $\alpha \geqslant 0$, it follows from (3.24) that

$$|E_{2N+2}| \leqslant K_4 \int_0^\infty u^{r+2N+2} e^{-(\lambda/2-K_2)u^2}\,du = 0(\lambda^{-(r+2N+3)/2}). \tag{3.31}$$

Furthermore, since both $D_{-r}(z)$ and $D'_{-r}(z)$ have no nonnegative real zeros, as long as $\lambda^{1/2}\alpha$ is bounded, say, $\lambda^{1/2}\alpha \leqslant A$, there exist positive numbers M_1 and M_2, independent of λ and α, such that

$$| e^{-\lambda\alpha^2/4}E_{2N+2} | \leqslant \lambda^{-(r+2N+3)/2}\{M_1 | D_{-r}(\lambda^{1/2}\alpha)| + M_2 | D'_{-r}(\lambda^{1/2}\alpha)|\}, \quad (3.32)$$

for sufficiently large λ. For $\lambda^{1/2}\alpha \geqslant A$, we have again from (3.24)

$$\begin{aligned} | E_{2N+2} | &\leqslant K_4 \int_0^\infty u^{r+2N+2}e^{-\lambda\alpha u}\, du \\ &= 0((\lambda\alpha)^{-r-2N-3}), \qquad \text{as} \quad \lambda \to \infty. \end{aligned} \quad (3.33)$$

Hence (3.32) holds also when $\lambda^{1/2}\alpha \geqslant A$, since

$$D_\nu(z) \sim z^\nu e^{-z^2/4}, \qquad D_\nu'(z) \sim (-\tfrac{1}{2})\, z^{\nu+1}e^{-z^2/4}, \quad (3.34)$$

as $z \to +\infty$ (see [8, p. 347]).

Using (3.28) and (3.32) it gives

$$\begin{aligned} e^{-\lambda\alpha^2/4}F(\lambda) = &\frac{D_{-r}(\lambda^{1/2}\alpha)}{\lambda^{r/2}}\left[\sum_{k=0}^{N} \frac{\zeta_k(\alpha)}{\lambda^k} + 0\left(\frac{1}{\lambda^{N+1}}\right)\right] \\ &+ \frac{D'_{-r}(\lambda^{1/2}\alpha)}{\lambda^{(r+1)/2}}\left[\sum_{k=0}^{N} \frac{\eta_k(\alpha)}{\lambda^k} + 0\left(\frac{1}{\lambda^{N+1}}\right)\right], \end{aligned} \quad (3.35)$$

which is the required result.

4. An Example

For each real number x and each positive integer n, set

$$e^{nx} = \sum_{n=0}^{n} \frac{(nx)^r}{r!} + \frac{(nx)^n}{n!}\, S_n(x). \quad (4.1)$$

In 1913, Ramanujan [5] asserted that, as $n \to \infty$,

$$S_n(1) = (n!/2)(e/n)^2 - \tfrac{2}{3} + (4/135n) + 0\,(1/n^2). \quad (4.2)$$

Proofs of this result were given independently in 1928 by Szego [6] and Watson [7]. In 1932, Copson [3] investigated the behaviour of $S_n(-1)$, and showed that, as $n \to \infty$,

$$S_n(-1) \sim -\tfrac{1}{2} + (1/8n) + (1/32n^2) - (1/128n^3) - (13/256n^4) + \cdots. \quad (4.3)$$

In a recent paper, Buckholtz [2] proved that, for $k \geqslant 1$

$$S_n(x) = \sum_{r=0}^{k-1} U_r(x) \frac{1}{n^r} + 0\left(\frac{1}{n^k}\right), \tag{4.4}$$

uniformly for $x \in (-\infty, 1 - \delta]$. The coefficients $U_r(x)$ are of the form

$$U_0(x) = x/(1 - x), \qquad U_r(x) = (-1)^r[Q_r(x)/(1 - x)^{2r+1}], \tag{4.5}$$

where $Q_r(x)$ is a polynomial in x of degree r. Since $U_r(x)$ has a pole of order $(2r + 1)$, the expansion (4.4) is not valid for x near 1. It is, therefore, natural to ask whether there exists an asymptotic expansion for $S_n(x)$, as $n \to \infty$, which is uniformly valid for x restricted to an interval, say, $\delta \leqslant x \leqslant 1$, where δ is some fixed positive number.

To answer this question affirmatively, we write

$$e^{nx} = \sum_{r=0}^{n} \frac{(nx)^r}{r!} + \frac{1}{n!} \int_0^{nx} (nx - t)^n e^t \, dt. \tag{4.6}$$

Comparing (4.1) with (4.6), we have

$$(nx)^n S_n(x) = e^{nx} \int_0^{nx} (nx - t)^n e^{-(nx-t)} \, dt. \tag{4.7}$$

Now make the substitution $(nx - t) = nx(1 - \tau)$ and write

$$S_n(x) = (nx) \int_0^1 \exp\{-n[-xt - \log(1 - t)]\} \, dt. \tag{4.8}$$

The last integral is clearly of the form (1.1), with $b = 1$, $r = 1$, $q(t) = 1$, $p(t) = [-xt - \log(1 - t)]$, $\lambda = n$ and $\alpha = (1 - x)/x$.

Following the procedures outlined in Section 2, we let

$$-xt - \log(1 - t) = (u^2/2) + au, \tag{4.9}$$

with

$$a^2 = 2\{(x - 1) - \log x\}. \tag{4.10}$$

Using $\mathfrak{P}(\beta)$ as a generic symbol for a power series in β such that $\mathfrak{P}(0) = 0$, we may express $a = a(x)$ in the form

$$a = (1 - x)[1 + \mathfrak{P}(1 - x)], \tag{4.11}$$

an analytic function of x near $x = 1$.

Changing to the variable u, we obtain

$$S_n(x) = (nx) \int_0^\infty \frac{dt}{du} e^{-n[u^2/2+au]} du, \tag{4.12}$$

where dt/du can be expanded as a convergent power series

$$\frac{dt}{du} = \sum_{n=0}^{\infty} a_n(x) u^n, \tag{4.13}$$

with the coefficients given by

$$a_n(x) = (1/n!)\{d^{n+1}t/du^{n+1}\}_{u=0} . \tag{4.14}$$

From (4.9) we have

$$[-x + (1/(1-t))](dt/du) = u + a, \tag{4.15}$$

and hence

$$\frac{1}{(1-t)^2}\left(\frac{dt}{du}\right)^2 + \left(-x + \frac{1}{1-t}\right)\frac{d^2t}{du^2} = 1,$$

$$\frac{2}{(1-t)^2}\left(\frac{dt}{du}\right)^3 + \frac{3}{(1-t)^2}\frac{dt}{du}\cdot\frac{d^2t}{du^2} + \left(-x + \frac{1}{1-t}\right)\frac{d^3t}{du^3} = 0 \tag{4.16}$$

and so on. These equations give

$$a_0(x) = a/(1-x), \qquad a_1(x) = [1/(1-x)] - [a^2/(1-x)^3],$$
$$a_2(x) = -[a^3/(1-x)^4] - [3a/2(1-x)^3] + [3a^3/2(1-x)^5], \tag{4.17}$$

and so on. Each coefficient is an analytic function of x near $x = 1$. Since $| dt/du |$ is bounded for large values of u, we have from the main theorem

$$e^{-na^2/4}S_n(x) \sim (nx)\left\{\frac{D_{-1}(n^{1/2}a)}{n^{1/2}} \sum_{k=0}^{\infty} \frac{\zeta_k(a)}{n^k} + \frac{D'_{-1}(n^{1/2}a)}{n} \sum_{k=0}^{\infty} \frac{\eta_k(a)}{n^k}\right\}, \tag{4.18}$$

as $n \to \infty$, where the coefficients $\zeta_k(a)$ and $\eta_k(a)$ are analytic functions of a near the origin. The dominant term of this expansion is

$$S_n(x) \sim [a/(1-x)]\, nx[D_{-1}(n^{1/2}a)/n^{1/2}]\, e^{na^2/4}, \qquad \text{as} \quad n \to \infty, \tag{4.19}$$

with $a^2 = 2\{(x-1) - \log x\}$.

As a check on the validity of (4.19), we first let $x = 1$. In this case we have

$$S_n(1) \sim (\pi n/2)^{1/2}, \qquad \text{as} \quad n \to \infty, \tag{4.20}$$

which is precisely the first approximation given by Watson. Next we let $x \leqslant 1 - \delta < 1$. Then we have

$$S_n(x) \sim x/(1 - x), \qquad \text{as} \quad n \to \infty, \tag{4.21}$$

which is the dominant term given by Buckholtz.

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